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Exact solutions to a nonlinear diffusion equation

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Abstract. Similarity solutions are derived for the nonlinear diffusion equation $c_t = \nabla \cdot (c\nabla c)$ in one dimension and with cylindrical symmetry. Some applications are indicated.

1. Introduction

In this paper we give further exact solutions of the type discussed in King [1]. We consider the radially symmetric nonlinear diffusion equation

$$\frac{\partial c}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} c \frac{\partial c}{\partial r}\right) \tag{1.1}$$

and seek similarity solutions of the form

$$c = t^{-N/(N+2)} f(r/t^{1/(N+2)}) \tag{1.2}$$

to obtain

$$-\frac{1}{N+2}\eta^N f = \eta^{N-1} f \frac{df}{d\eta} + \alpha \tag{1.3}$$

where $\eta = r/t^{1/(N+2)}$, and α is an arbitrary constant of integration. When $\alpha = 0$, the solution to (1.3) is easily determined in closed form. Here we shall obtain the general solution to (1.3) for $\alpha \neq 0$ when $N = 1$ and when $N = 2$. It is convenient to first introduce

$$g = f + \frac{\eta^2}{2(N+2)}$$

and to rewrite (1.3) as

$$\alpha \frac{d\eta}{dg} = \eta^{N-1}\left(\frac{\eta^2}{2(N+2)} - g\right). \tag{1.4}$$

2. Exact solutions

2.1. $N = 1$

Equation (1.4) is then the Riccati equation

$$\alpha \frac{d\eta}{dg} = \frac{\eta^2}{6} - g. \tag{2.1}$$

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Writing

$$\eta = -6\alpha \frac{dq}{dg} q^{-1}$$

we obtain the linear equation

$$6\alpha^2 \frac{d^2q}{dg^2} = gq$$

with general solution

$$q = a\ \mathrm{Ai}((6\alpha^2)^{-1/3}g) + b\ \mathrm{Bi}((6a^2)^{-1/3}g)$$

where a and b are arbitrary constants, and Ai and Bi are Airy functions. We therefore have

$$\eta = -(36\alpha)^{1/3}\left(\frac{\sin\gamma\ \mathrm{Ai}'((6\alpha^2)^{-1/3}(f+\frac{1}{6}\eta^2)) + \cos\gamma\ \mathrm{Bi}'((6\alpha^2)^{-1/3}(f+\frac{1}{6}\eta^2))}{\sin\gamma\ \mathrm{Ai}((6\alpha^2)^{-1/3}(f+\frac{1}{6}\eta^2)) + \cos\gamma\ \mathrm{Bi}((6\alpha^2)^{-1/3}(f+\frac{1}{6}\eta^2))}\right) \tag{2.2}$$

where $\sin\gamma = a/(a^2+b^2)^{1/2}$.

Later we shall also need the solution of the form

$$c = (-t)^{-1/3} f(r/(-t)^{1/3}). \tag{2.3}$$

We then have

$$\tfrac{1}{3}\eta f = f\frac{df}{d\eta} + \alpha$$

where $\eta = r/(-t)^{1/3}$ and the solution (which can be obtained from (2.2) by replacing f by $-f$) is

$$\eta = -(36\alpha)^{1/3}\left(\frac{\sin\gamma\ \mathrm{Ai}'((6\alpha^2)^{-1/3}(\frac{1}{6}\eta^2-f)) + \cos\gamma\ \mathrm{Bi}'((6\alpha^2)^{-1/3}(\frac{1}{6}\eta^2-f))}{\sin\gamma\ \mathrm{Ai}((6\alpha^2)^{-1/3}(\frac{1}{6}\eta^2-f)) + \cos\gamma\ \mathrm{Bi}((6\alpha^2)^{-1/3}(\frac{1}{6}\eta^2-f))}\right). \tag{2.4}$$

2.2. N=2

Equation (1.4) is then the Bernoulli equation

$$\alpha \frac{d\eta}{dg} = \frac{\eta^3}{8} - \eta g.$$

The solution, which is a special case of one derived in [2] (though its application to the cylindrically symmetric problem was not noted there) is obtained by writing $\eta = \xi^{-1/2}$ to give

$$\alpha \frac{d\xi}{dg} = 2\xi g - \tfrac{1}{4}$$

so that

$$\xi = e^{g^2/\alpha}(\tfrac{1}{8}\sqrt{\pi/\alpha}\ \mathrm{erfc}(g/\sqrt{\alpha}) + \beta)$$

where β is an arbitrary constant.

Hence we obtain

$$\eta = \exp(-(f+\tfrac{1}{8}\eta^2)^2/2\alpha)\{\tfrac{1}{8}\sqrt{\pi/\alpha}\ \mathrm{erfc}((f+\tfrac{1}{8}\eta^2)/\sqrt{\alpha}) + \beta\}^{-1/2}. \tag{2.5}$$

The solution (2.5) satisfies

$$f \sim (\alpha \ln(1/\beta\eta^2))^{1/2} \qquad \text{as } \eta \to 0$$

and this singularity makes the solution harder to interpret than that of section 2.1 (it does, however, correspond to a finite flux across $r=0$). In the next section we shall briefly discuss possible applications of (2.2) and (2.4).

3. Applications

Here we consider two simple problems whose solutions can be determined from the results of section 2.1.

Replacing r by x, we are considering

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c\frac{\partial c}{\partial x}\right). \tag{3.1}$$

We first discuss the initial-boundary value problem for (3.1) subject to

$$\begin{aligned} &c = 0 && \text{at } x = 0 \\ &c \to 0 && \text{at } x \to +\infty \\ &c = D/x && \text{at } t = 0 \end{aligned}$$

for some constant $D > 0$. The integral result

$$\int_0^\infty (xc - D)\,\mathrm{d}x = 0$$

then holds for all t and the solution takes the form

$$c = t^{-1/3} f(x/t^{1/3}) \tag{3.2}$$

where f is given by (2.2) with $\eta = x/t^{1/3}$, $a = -\frac{1}{3}D$ and $\gamma = \pi/3$; the value of γ follows from $f(0) = 0$ and $\mathrm{Bi}'(0)/\mathrm{Ai}'(0) = -\sqrt{3}$.

This solution will also describe the large-time behaviour for more general initial conditions such that $c \sim D/x$ as $x \to +\infty$. It might at first appear that the similarity form (3.2) is also appropriate for describing the large-time behaviour for (3.1) subject to

$$\begin{aligned} &\partial c/\partial x = 0 && \text{at } x = 0 \\ &c \to 0 && \text{as } x \to +\infty \\ &c = c_0(x) && \text{at } t = 0 \end{aligned} \tag{3.3}$$

where $c_0 \sim D/x$ as $x \to +\infty$. However, the condition on $x = 0$ cannot be satisfied unless $\alpha = 0$, which requires $D = 0$. It follows from the results of [3] that the large-time behaviour for (3.1) subject to (3.3) in fact takes the form

$$c \sim t^{-1/3} \ln^{2/3} t\, f(x/t^{1/3} \ln^{1/3} t)$$

rather than (3.2).

As our second application, we apply (2.4) to a peaking problem (see, for example, [4] and [5]). We solve (3.1) for $t < 0$ subject to

$$\begin{aligned} &c = D(-t)^{-1/3} && \text{at } x = 0 \\ &c \to 0 && \text{as } x \to +\infty \\ &c \to 0 && \text{at } t \to -\infty. \end{aligned}$$

The solution takes the form (2.3) and in (2.4) we require $f \to 0$ as $\eta \to +\infty$ which may be shown to give $\gamma = \pi/2$, so that

$$\eta = -(36a)^{1/3}\,\mathrm{Ai}'((6\alpha^2)^{-1/3}(\tfrac{1}{6}\eta^2 - f))/\mathrm{Ai}((6\alpha^2)^{-1/3}(\tfrac{1}{6}\eta^2 - f)) \qquad (3.4)$$

which implies that

$$f \sim 3\alpha/\eta$$

as $\eta \to +\infty$.

We determine α from the condition that $f(0) = D$ which requires

$$\mathrm{Ai}'(-(6\alpha^2)^{-1/3}D) = 0.$$

We need the first zero of Ai′ which gives

$$(6\alpha^2)^{-1/3}D = 1.018\,79\ldots$$

(see [6]).

4. Discussion

In [1] we derived general solutions to a number of similarity ordinary differential equations for power law diffusivities, and for $N = 1$, 2 and 3 these all correspond to negative powers (the 'fast' diffusion case). Here we have obtained two general solutions for the particular 'slow' diffusion case (1.1). It should be noted that some of the discussion in sections 3.1 and 6 of [1] concerning the behaviour of the 'dipole' solutions implicitly assumed that $(nN+2)/(n+1) > 0$ (in the notation of that paper); when this condition is violated the discussion is incorrect in places.

Some exact solutions for 'fast' diffusion with peaking can fairly easily be found; for example, if we consider

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^{-m}\frac{\partial c}{\partial x}\right)$$

$$c = D(-t)^{-1/m} \qquad \text{at } x = 0$$

$$c \to 0 \text{ with } c^{-m}(\partial c/\partial x) \to 0 \qquad \text{as } x \to +\infty$$

$$c \to 0 \qquad \text{at } t \to -\infty$$

for $t < 0$, then the solution is given for $m > 0$ by the travelling wave

$$c = D(m^{1/2}D^{m/2}x - t)^{-1/m}.$$

Exact solutions for the 'slow' case are harder to find, but one is given by (3.4).

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